ATKINSON SCIENCE

THEORY GUIDE

Admittance Method

2 Mathematical Development

Keith Atkinson

23 October 2020

Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

Contents

1 Introduction 5
2 Temperature excitation on one side of a homogeneous slab 7
3 Temperature excitation on the reverse side of the slab 13
4 Temperature excitation on both sides of the slab 17
5 Cyclic surface temperature variation 23
6 Inverse complex transmission matrix 35
7 Temperature excitation on both sides of the slab 41
8 References 49
9 Appendix 51
9.1 Transpose of a matrix 51
9.2 Determinant of a matrix 51
9.3 Unit matrix 54
9.4 Inverse of a matrix 54
9.5 Inverse of the complex transmission matrix for a uniform slab 56

Figures

Figure 1 Conducting slab with temperature excitation on the side $x = 0$ 7
Figure 2 Conducting slab with temperature excitation on right side 13
Figure 3 Conducting slab with temperature excitation on both sides 17
Figure 4 Amplitude and phase of $N/\sinh M$ 27
Figure 5 Amplitude and phase of $N/\sinh M$ 29
Figure 6 Amplitude and phase of $N/\tanh M$ 32
Figure 7 Amplitude and phase of $-N/\sinh M$ 36
Figure 8 Temperature θ at $x = L$ and heat flux q at $x = 0$ 38
Figure 9 Amplitude and phase of $-N/\tanh M$ 39
Figure 10 Heat fluxes due to sinusoidal temperature oscillations at $x = 0$ and $x = L$ 43
Figure 11 Temperature θ at $x = 0$ and heat flux q at $x = 0$ and $x = L$ 45
Figure 12 Temperature θ at $x = L$ and heat flux q at $x = 0$ and $x = L$ 47
Figure 13 Net heat flux q at $x = 0$ and $x = L$ 48

1 Introduction

In the previous report in this series, Ref. [1], we introduced Fourier's law of heat conduction,

$$q(x,t) = -\,k(x)\frac{\partial\theta(x,t)}{\partial x} \qquad (1.1)$$

where $q(x,\,t)$ [W m^{-2}] is the heat flux in the direction x [m] normal to a flat plate of infinitesimal thickness, with thermal conductivity $k(x)$ [W m^{-1} K^{-1}], and $\theta(x,\,t)$ [K] is the temperature in the plate.

By using Fourier's law we derived the equation of one-dimensional transient heat conduction for a material whose thermal properties vary in the heat flow direction:

$$\rho(x)C(x)\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[k(x)\frac{\partial\theta}{\partial x}\right] \qquad (1.2)$$

The thermal properties ρ [kg m^{-3}] and C [J kg^{-1} K^{-1}] are the density and specific heat capacity of the material, respectively. If the thermal properties are constant, then (1.2) simplifies to

$$\frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2} \qquad (1.3)$$

where $\alpha = k/(\rho C)$ [m^2 s^{-1}] is the (constant) thermal diffusivity of the material.

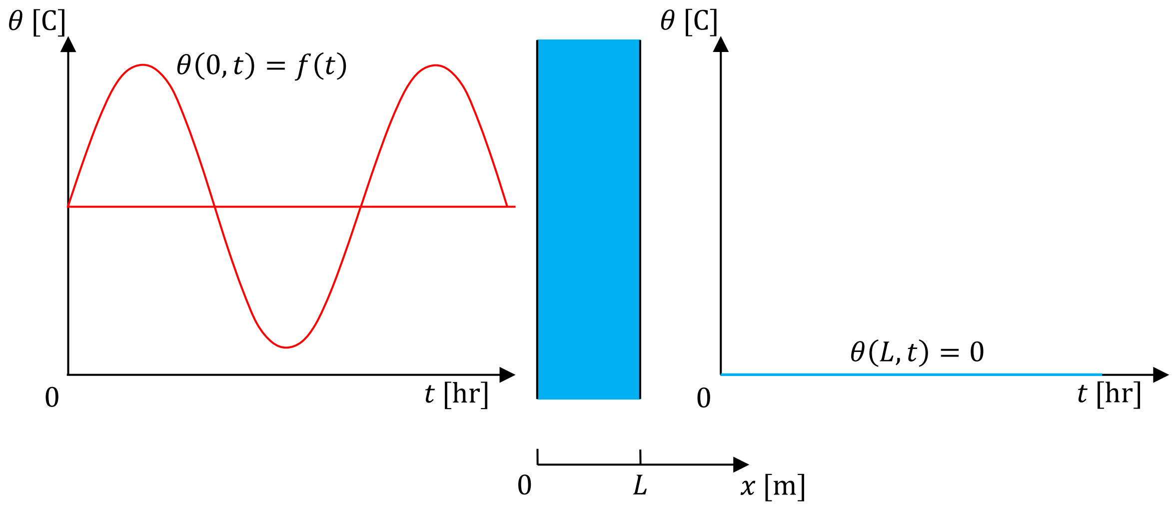
Finally, we showed how we can use (1.3) to calculate the time-varying temperature through a homogenous slab when the temperature on one side of the slab is made to oscillate sinusoidally and the temperature on the other side is held constant. In this report we shall focus on calculating the time-varying heat fluxes into and out of the slab and develop the method known as the *admittance method*.

Sections 2 to 7 present the mathematical analysis that underlies the admittance method. The important practical results from the analysis are summarised in Section 7. All of the mathematics in this report is covered in most textbooks on advanced mathematics for engineers and scientists. Ref. [2] covers all of the mathematics in the complete series of reports. This report contains numerous worked examples to help the reader become familiar with and use the admittance method.

2 Temperature excitation on one side of a homogeneous slab

We shall begin the mathematical analysis by considering a slab of homogeneous conducting material with thickness L [m], as shown in Figure 1.

Figure 1 Conducting slab with temperature excitation on the side $x = 0$



We want to calculate the instantaneous heat flux $q(x, t)$ [W m^{-2}] in the slab, given the initial condition

$$\theta(x, 0) \text{ [K]} = 0 \qquad (2.1)$$

and boundary conditions

$$\theta(0, t) = f(t) \qquad (2.2)$$

$$\theta(L, t) = 0 \qquad (2.3)$$

We can solve this boundary-value problem using Laplace transforms. We first take the Laplace transform of the heat conduction equation (1.3) and associated boundary conditions with respect to one of the independent variables. We then have an ordinary differential equation (ODE) for the Laplace transform of the required solution. We solve the ODE to obtain the Laplace transform. However, we shall not attempt to find the inverse Laplace transform and thus the temperature in the slab.

We shall take the Laplace transform of (1.3) with respect to t. The Laplace transform of the left hand side of (1.3) is:

$$\mathcal{L}\{\theta'(x,t)\} = s\mathcal{L}\{\theta(x,t)\} - \theta(x,0)$$

$$= s\int_0^\infty e^{-st}\theta\,dt - \theta(x,0)$$

$$= s\Theta(x,s) - \theta(x,0)$$

The Laplace transform of the right hand side is:

$$\mathcal{L}\left\{\alpha\frac{\partial^2\theta}{\partial x^2}\right\} = \int_0^\infty e^{-st}\left(\alpha\frac{\partial^2\theta}{\partial x^2}\right)dt$$

$$= \alpha\frac{d^2}{dx^2}\int_0^\infty e^{-st}\theta\,dt$$

$$= \alpha\frac{d^2\Theta}{dx^2}$$

Equating the left and right hand sides gives

$$s\Theta - \theta(x,0) = \alpha\frac{d^2\Theta}{dx^2}$$

or

$$\frac{d^2\Theta}{dx^2} = \frac{s}{\alpha}\Theta - \frac{1}{\alpha}\theta(x,0) \qquad (2.4)$$

and applying the initial condition (2.1) gives

$$\frac{d^2\Theta}{dx^2} = \frac{s}{\alpha}\Theta \qquad (2.5)$$

Taking the Laplace transforms of the boundary conditions (2.2) and (2.3), we have

$$\Theta(0,s) = \mathcal{L}\{\theta(0,t)\} = \mathcal{L}\{f(t)\} = F(s) \qquad (2.6)$$

$$\Theta(L,s) = \mathcal{L}\{\theta(L,t)\} = \mathcal{L}\{0\} = 0 \qquad (2.7)$$

We now have an ordinary differential equation (2.5) and associated boundary conditions (2.6) and (2.7) for the Laplace transform of the required solution.

Equation (2.5) is a homogeneous linear ordinary differential equation with constant coefficients, so we can use the elementary methods set out in Ref. [2]. The order n of the equation is 2, so we must find two linearly independent solutions of the equation. Let $\Theta = e^{mx}$ where m is a constant. Substituting into (2.5) gives

$$\left(m^2 - \frac{s}{\alpha}\right) e^{mx} = 0$$

or

$$m^2 - \frac{s}{\alpha} = 0 \qquad (2.8)$$

The roots m of (2.8) are $\pm\sqrt{s/\alpha}$, and the required linearly independent solutions of (2.5) are

$$\Theta_1 = e^{x\sqrt{s/\alpha}} \text{ and } \Theta_2 = e^{-x\sqrt{s/\alpha}}$$

The general solution of (2.5) is therefore

$$\Theta = c_1\Theta_1 + c_2\Theta_2$$

$$\Theta(x,s) = c_1 e^{x\sqrt{s/\alpha}} + c_2 e^{-x\sqrt{s/\alpha}} \qquad (2.9)$$

Applying the boundary conditions (2.6) and (2.7) to (2.9) gives

$$F(s) = c_1 e^{(0)\sqrt{s/\alpha}} + c_2 e^{-(0)\sqrt{s/\alpha}} = c_1 + c_2 \qquad (2.10)$$

$$0 = c_1 e^{L\sqrt{s/\alpha}} + c_2 e^{-L\sqrt{s/\alpha}} \qquad (2.11)$$

We can solve the simultaneous equations (2.10) and (2.11) as follows. From (2.10),

$$c_2 = F(s) - c_1$$

Substituting into (2.11) gives

$$c_1 e^{L\sqrt{s/\alpha}} + (F(s) - c_1)e^{-L\sqrt{s/\alpha}} = 0$$

or

$$c_1 = \frac{-F(s)e^{-L\sqrt{s/\alpha}}}{e^{L\sqrt{s/\alpha}} - e^{-L\sqrt{s/\alpha}}} \qquad (2.12)$$

Then, we have

$$c_2 = F(s) - c_1$$

$$= F(s) + \frac{-F(s)e^{-L\sqrt{s/\alpha}}}{e^{L\sqrt{s/\alpha}} - e^{-L\sqrt{s/\alpha}}}$$

$$= \frac{F(s)e^{L\sqrt{s/\alpha}}}{e^{L\sqrt{s/\alpha}} - e^{-L\sqrt{s/\alpha}}} \qquad (2.13)$$

Finally, substituting (2.12) and (2.13) into (2.9) gives

$$\Theta(x,s) = \frac{-F(s)e^{-L\sqrt{s/\alpha}}\, e^{x\sqrt{s/\alpha}} + F(s)e^{L\sqrt{s/\alpha}}\, e^{-x\sqrt{s/\alpha}}}{e^{L\sqrt{s/\alpha}} - e^{-L\sqrt{s/\alpha}}}$$

$$= \frac{e^{(L-x)\sqrt{s/\alpha}} - e^{-(L-x)\sqrt{s/\alpha}}}{e^{L\sqrt{s/\alpha}} - e^{-L\sqrt{s/\alpha}}} F(s)$$

$$= \frac{\sinh\left((L-x)\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)} F(s) \qquad (2.14)$$

We shall not attempt to find the inverse Laplace transform of (2.14) and thus the temperature $\theta(x\,,t)$.

We can also transform Fourier's law:

$$q(x,t) = -k\frac{\partial\theta(x,t)}{\partial x} \qquad (2.15)$$

into the Laplace domain. The Laplace transform of the left hand side of (2.15) is

$$\mathcal{L}\{q(x,t)\} = Q(x,s) \qquad (2.16)$$

and the Laplace transform of the right hand side of (2.15) is

$$\mathcal{L}\left\{-k\frac{\partial\theta(x,t)}{\partial x}\right\} = \int_0^\infty e^{-st}\left(-k\frac{\partial\theta(x,t)}{\partial x}\right)dt$$

$$= -k\frac{d}{dx}\int_0^\infty e^{-st}\theta(x,t)\,dt$$

$$= -k\frac{d}{dx}\Theta(x,s) \qquad (2.17)$$

Equating the left and right hand sides gives

$$Q(x,s) = -k\frac{d}{dx}\Theta(x,s) \qquad (2.18)$$

We already have $\Theta(x, s)$ from Equation (2.14). Substituting (2.14) into (2.18) gives

$$Q(x,s) = -k\frac{d}{dx}\left[\frac{\sinh\left((L-x)\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}F(s)\right]$$

$$= -k\left[\frac{-\sqrt{s/\alpha}\cosh\left((L-x)\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}F(s)\right]$$

$$= k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left((L-x)\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right]F(s) \qquad (2.19)$$

We now have a solution in the Laplace domain of the heat flux in the slab due to a temperature excitation on the side $x = 0$ of the slab. We shall return to this solution in Section 4.

3 Temperature excitation on the reverse side of the slab

We now want to solve the heat conduction equation with the initial condition

$$\theta(x,0) = 0 \qquad (3.1)$$

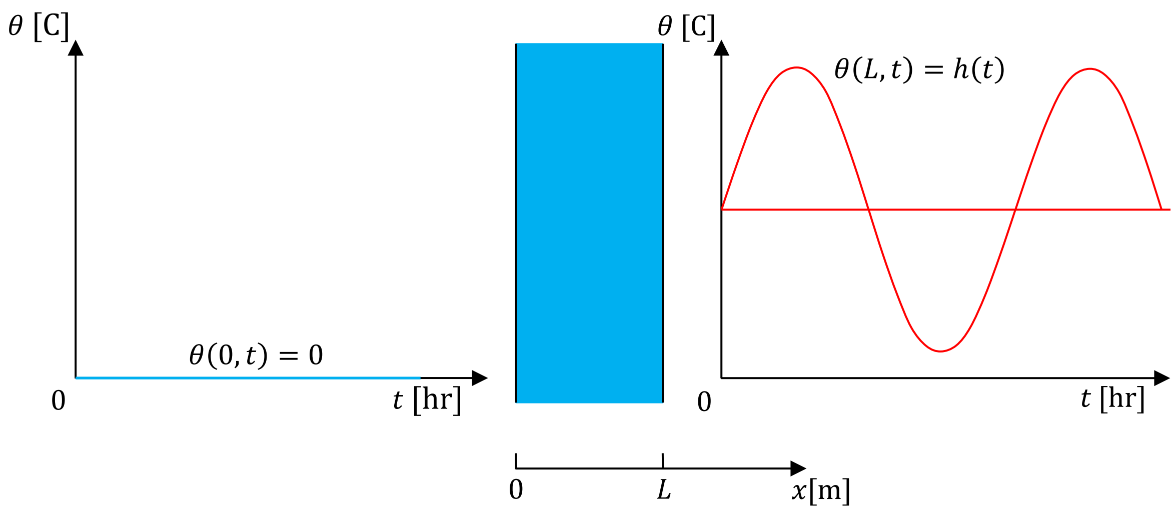
and the boundary conditions

$$\theta(0,t) = 0 \qquad (3.2)$$

$$\theta(L,t) = h(t) \qquad (3.3)$$

as shown in Figure 2.

Figure 2 Conducting slab with temperature excitation on right side



Recall the heat conduction equation (1.3):

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \qquad (1.3)$$

We showed that the Laplace transform of (1.3) is given by (2.4):

$$\frac{d^2 \Theta(x,s)}{dx^2} = \frac{s}{\alpha}\Theta(x,s) - \frac{1}{\alpha}\theta(x,0) \qquad (2.4)$$

Applying the initial condition (3.1) gives

$$\frac{d^2 \Theta}{dx^2} = \frac{s}{\alpha}\Theta \qquad (3.4)$$

Taking the Laplace transforms of the boundary conditions (3.2) and (3.3) gives

$$\Theta(0,s) = \mathcal{L}\{\theta(0,t)\} = \mathcal{L}\{0\} = 0 \quad (3.5)$$

$$\Theta(L,s) = \mathcal{L}\{\theta(L,t)\} = \mathcal{L}\{h(t)\} = H(s) \quad (3.6)$$

In Section 1 we showed that the general solution of (2.4) is

$$\Theta(x,s) = c_1 e^{x\sqrt{s/\alpha}} + c_2 e^{-x\sqrt{s/\alpha}} \quad (3.7)$$

By applying the boundary conditions (3.5) and (3.6) to (3.4), we have

$$0 = c_1 e^{(0)\sqrt{s/\alpha}} + c_2 e^{-(0)\sqrt{s/\alpha}} = c_1 + c_2 \quad (3.8)$$

$$H(s) = c_1 e^{L\sqrt{s/\alpha}} + c_2 e^{-L\sqrt{s/\alpha}} \quad (3.9)$$

We can solve the simultaneous equations (3.8) and (3.9) as follows. From (3.8),

$$c_2 = -c_1$$

Substituting into (3.9) gives

$$c_1 e^{L\sqrt{s/\alpha}} - c_1 e^{-L\sqrt{s/\alpha}} = H(s)$$

or

$$c_1 = \frac{H(s)}{e^{L\sqrt{s/\alpha}} - e^{-L\sqrt{s/\alpha}}} \quad (3.10)$$

Then, we have

$$c_2 = -c_1 = \frac{-H(s)}{e^{L\sqrt{s/\alpha}} - e^{-L\sqrt{s/\alpha}}} \quad (3.11)$$

Finally, substituting (3.10) and (3.11) into (3.7) gives

$$\Theta(x,s) = \frac{H(s)e^{x\sqrt{s/\alpha}} + H(s)e^{-x\sqrt{s/\alpha}}}{e^{L\sqrt{s/\alpha}} - e^{-L\sqrt{s/\alpha}}}$$

$$= \frac{e^{x\sqrt{s/\alpha}} - e^{-x\sqrt{s/\alpha}}}{e^{L\sqrt{s/\alpha}} - e^{-L\sqrt{s/\alpha}}} H(s)$$

$$= \frac{\sinh\left(x\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)} H(s) \quad (3.12)$$

We shall not attempt to find the inverse Laplace transform of (3.12) and thus the temperature $\theta(x,t)$.

We have already calculated the Laplace transform of Fourier's law. Recall Equation (2.18):

$$Q(x,s) = -k\frac{d}{dx}\Theta(x,s) \qquad (2.18)$$

We have $\Theta(x,s)$ from Equation (3.12). Substituting (3.12) into (2.18) gives

$$Q(x,s) = -k\frac{d}{dx}\left[\frac{\sinh\left(x\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}H(s)\right]$$

$$= -k\left[\frac{\sqrt{s/\alpha}\cosh\left(x\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}H(s)\right]$$

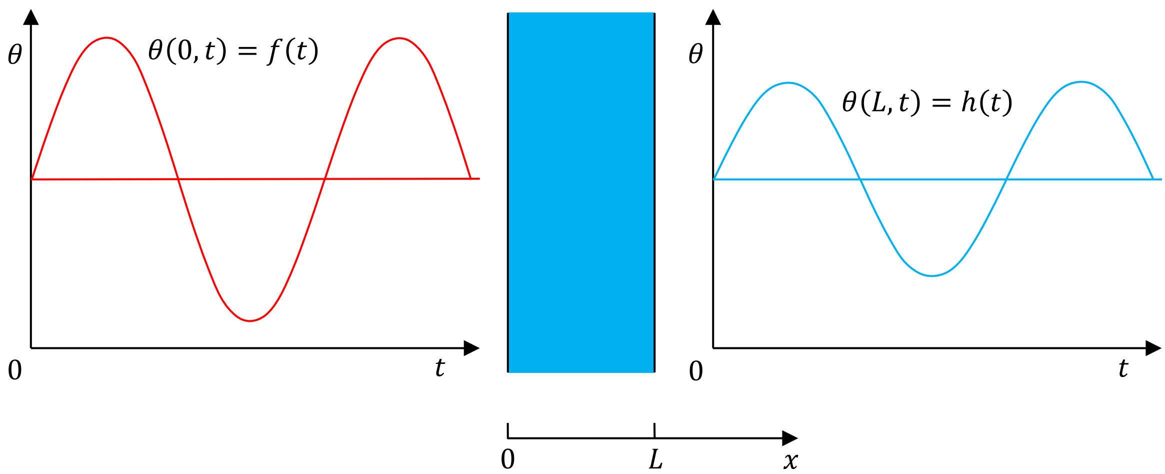
$$= -k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left(x\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right]H(s) \qquad (3.13)$$

We now have a solution in the Laplace domain of the heat flux in the slab due to a temperature excitation on the side $x = L$ of the slab. We shall return to this solution in Section 4.

4 Temperature excitation on both sides of the slab

We now want to solve the heat conduction equation with excitation on both sides of the slab, as shown in Figure 3.

Figure 3 Conducting slab with temperature excitation on both sides



We have calculated the heat flux in the Laplace domain subject to temperature excitations $F(s)$ at $x = 0$. Recall Equation (2.19):

$$Q(x,s) = k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left((L-x)\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right]F(s) \qquad (2.19)$$

At $x = 0$,

$$Q(0,s) = k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left(L\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right]F(s)$$

$$= U(s)F(s) \qquad (4.1)$$

where

$$U(s) = k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left(L\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right] \qquad (4.2)$$

At $x = L$ [m],

$$Q(L,s) = k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left((0)\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right]F(s)$$

$$= V(s)F(s) \qquad (4.3)$$

where

$$V(s) = k\sqrt{\frac{s}{\alpha}}\left[\frac{1}{\sinh\left(L\sqrt{s/\alpha}\right)}\right] \qquad (4.4)$$

We have also calculated the heat flux in the Laplace domain subject to temperature excitations $H(s)$ at $x = L$ [m]. Recall Equation (3.13):

$$Q(x,s) = -k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left(x\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right]H(s) \qquad (3.13)$$

At $x = 0$,

$$Q(0,s) = -k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left((0)\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}L\right)}\right]H(s)$$

$$= W(s)H(s) \qquad (4.5)$$

where

$$W(s) = -k\sqrt{\frac{s}{\alpha}}\left[\frac{1}{\sinh\left(L\sqrt{s/\alpha}\right)}\right] \qquad (4.6)$$

At $x = L$ [m],

$$Q(L,s) = -k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left(L\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right]H(s)$$

$$= X(s)H(s) \qquad (4.7)$$

where

$$X(s) = -k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left(L\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right] \qquad (4.8)$$

We can add the two solutions to determine the heat flux in the Laplace domain subject to the temperature excitations $F(s)$ at $x = 0$ and the temperature excitations $H(s)$ at $x = L$ [m].

To determine the heat flux at $x = 0$ we add equations (4.1) and 4.5):

$$Q(0, s) = U(s)F(s) + W(s)H(s) \quad (4.9)$$

To determine the heat flux at $x = L$ m we add equations (4.3) and (4.7):

$$Q(L, s) = V(s)F(s) + X(s)H(s) \quad (4.10)$$

Equations (4.9) and (4.10) can be written in the following matrix form:

$$\begin{bmatrix} Q(0, s) \\ Q(L, s) \end{bmatrix} = \begin{bmatrix} U(s) & W(s) \\ V(s) & X(s) \end{bmatrix} \begin{bmatrix} F(s) \\ H(s) \end{bmatrix} \quad (4.11)$$

Using equation (4.11), we can determine the heat fluxes $Q(0, s)$ and $Q(L, s)$ on the two sides of the layer (the "output") in response to the temperature excitations $F(s)$ and $H(s)$ on the two sides of the wall (the "input").

The functions $F(s)$ and $H(s)$ are both Laplace transforms of temperature functions. If we invert the coefficient matrix in (4.11) we can determine the temperatures $F(s)$ and $H(s)$ on the two sides of the layer in response to the heat fluxes on the two sides of the layer:

$$\begin{bmatrix} F(s) \\ H(s) \end{bmatrix} = \begin{bmatrix} U(s) & W(s) \\ V(s) & X(s) \end{bmatrix}^{-1} \begin{bmatrix} Q(0, s) \\ Q(L, s) \end{bmatrix} \quad (4.12)$$

By rearranging (4.10), we obtain

$$F(s) = -\frac{X(s)}{V(s)} H(s) + \frac{1}{V(s)} Q(L, s) \quad (4.13)$$

Eliminating $F(s)$ between (4.9) and (4.10) gives

$$Q(0, s) = U(s) \left(\frac{Q(L, s) - X(s)H(s)}{V(s)} \right) + W(s)H(s)$$

$$= \left(W(s) - \frac{U(s)X(s)}{V(s)} \right) H(s) + \frac{U(s)}{V(s)} Q(L, s) \quad (4.14)$$

Equations (4.13) and (4.14) can be written in the following matrix form:

$$\begin{bmatrix} F(s) \\ Q(0, s) \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} H(s) \\ Q(L, s) \end{bmatrix} \quad (4.15)$$

where

$$A(s) = -\frac{X(s)}{V(s)}$$

$$B(s) = \frac{1}{V(s)}$$

$$C(s) = W(s) - \frac{U(s)X(s)}{V(s)}$$

$$D(s) = \frac{U(s)}{V(s)}$$

Substituting the equations (4.2), (4.4), (4.6), (4.8) for $U(s)$, $V(s)$, $W(s)$, $X(s)$, respectively,

$$A(s) = -\frac{X(s)}{V(s)} = \cosh\left(L\sqrt{s/\alpha}\right) \qquad (4.16)$$

$$B(s) = \frac{1}{V(s)} = \frac{\sinh\left(L\sqrt{s/\alpha}\right)}{k\sqrt{s/\alpha}} \qquad (4.17)$$

$$C(s) = W(s) - \frac{U(s)X(s)}{V(s)}$$

$$= -k\sqrt{\frac{s}{\alpha}}\left[\frac{1}{\sinh\left(L\sqrt{s/\alpha}\right)}\right] + \frac{k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left(L\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right] k\sqrt{\frac{s}{\alpha}}\left[\frac{\cosh\left(L\sqrt{s/\alpha}\right)}{\sinh\left(L\sqrt{s/\alpha}\right)}\right]}{k\sqrt{\frac{s}{\alpha}}\left[\frac{1}{\sinh\left(L\sqrt{s/\alpha}\right)}\right]}$$

$$= -k\sqrt{\frac{s}{\alpha}}\frac{\left[1 - \cosh^2\left(L\sqrt{s/\alpha}\right)\right]}{\sinh\left(L\sqrt{s/\alpha}\right)}$$

$$= -k\sqrt{\frac{s}{\alpha}}\frac{\left[-\sinh^2\left(L\sqrt{s/\alpha}\right)\right]}{\sinh\left(L\sqrt{s/\alpha}\right)}$$

$$= k\sqrt{s/\alpha}\,\sinh\left(L\sqrt{s/\alpha}\right) \qquad (4.18)$$

$$D(s) = \frac{U(s)}{V(s)} = \cosh\left(L\sqrt{s/\alpha}\right) \qquad (4.19)$$

Thus Equation (4.15) can be written

$$\begin{bmatrix} F(s) \\ Q(0,s) \end{bmatrix} = \begin{bmatrix} \cosh\left(L\sqrt{s/\alpha}\right) & \dfrac{\sinh\left(L\sqrt{s/\alpha}\right)}{k\sqrt{s/\alpha}} \\ k\sqrt{s/\alpha}\,\sinh\left(L\sqrt{s/\alpha}\right) & \cosh\left(L\sqrt{s/\alpha}\right) \end{bmatrix} \begin{bmatrix} H(s) \\ Q(L,s) \end{bmatrix}$$

$$= \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N\sinh M & \cosh M \end{bmatrix} \begin{bmatrix} H(s) \\ Q(L,s) \end{bmatrix} \qquad (4.20)$$

where

$$M = L\sqrt{s/\alpha} \qquad (4.21)$$

and

$$N = k\sqrt{s/\alpha} \qquad (4.22)$$

We now have a matrix equation in the Laplace domain relating the temperature forcing terms $F(s)$ and $H(s)$ to the heat flux terms $Q(0, s)$ and $Q(L, s)$. In the next section we shall rewrite the equation in terms of the frequency variable $j\omega$.

5 Cyclic surface temperature variation

In problems with cyclic surface temperature variations it is necessary to modify the constants M and N given by equations (4.21) and (4.22), by replacing the Laplace domain variable s with $j\omega$, where j is the imaginary constant $\sqrt{-1}$, ω [rad s^{-1}] is the angular frequency $2\pi f$, and f (cycles per second) is the frequency of the temperature variations:

$$M = L\sqrt{j\omega/\alpha} \qquad (5.1)$$

and

$$N = k\sqrt{j\omega/\alpha} \qquad (5.2)$$

Equation (4.20) becomes

$$\begin{bmatrix} F(j\omega) \\ Q(0, j\omega) \end{bmatrix} = \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{bmatrix} \begin{bmatrix} H(j\omega) \\ Q(L, j\omega) \end{bmatrix} \qquad (5.3)$$

where $F(j\omega)$ and $H(j\omega)$ are complex temperatures and $Q(0, j\omega)$ and $Q(L, j\omega)$ are complex heat fluxes. The square matrix in (5.3) is the *complex transmission matrix* of the slab.

To apply the complex transmission matrix to the heating of buildings, we assume that the temperature variation on the face $x = 0$ of the slab is

$$F(j\omega) = A_0 \sin(\omega t) = \mathrm{Im}\left(A_0 e^{j\omega t}\right) \qquad (5.4)$$

where A_0 is the amplitude of the temperature and Im means "the imaginary part of". The temperature on the opposite face $x = L$ [m] is maintained at zero. If we replace $F(j\omega)$ in (5.3) with A_0 then (5.3) reduces to

$$\begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} = \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{bmatrix} \begin{bmatrix} 0 \\ Q(L) \end{bmatrix} \qquad (5.5)$$

where $Q(0)$ and $Q(L)$ are complex constants. In Equation (5.5) A_0 is used as a reference temperature and the phases of all the other quantities are determined with respect to the temperature $A_0 \sin(\omega t)$.

From (5.5) we obtain:

$$A_0 = \frac{\sinh M}{N} Q(L)$$

and

$$Q(0) = Q(L) \cosh M$$

From these two equations we obtain

$$Q(0) = A_0 \frac{N}{\tanh M} \qquad (5.6)$$

and

$$Q(L) = A_0 \frac{N}{\sinh M} \qquad (5.7)$$

The instantaneous heat flux through the face at $x = 0$ is

$$q(0, t) = \mathrm{Im}\big[Q(0)e^{j\omega t}\big]$$

$$= \mathrm{Im}\left[A_0 \frac{N}{\tanh M} e^{j\omega t}\right] \qquad (5.8)$$

and the instantaneous heat flux through the face at $x = L$ is

$$q(L, t) = \mathrm{Im}\big[Q(L)e^{j\omega t}\big]$$

$$= \mathrm{Im}\left[A_0 \frac{N}{\sinh M} e^{j\omega t}\right] \qquad (5.9)$$

Equations (5.4), (5.8) and (5.9) enable us to specify a sinusoidal temperature oscillation on the face at $x = 0$ of a homogeneous slab and to determine the resulting heat fluxes on the faces at $x = 0$ and $x = L$. In the following worked examples the equations are applied to some practical problems.

Example 1

A wall is made of bricks with a thickness L of 0.105 m. The density ρ, specific heat capacity C, and thermal conductivity k of the bricks are 1,700 kg m^{-3}, 800 J kg^{-1} K^{-1} and 0.84 W m^{-1} K^{-1}, respectively. Calculate the transmission matrix in Equation (5.5) for the wall.

The diffusivity α [m^2 s^{-1}] of the bricks is

$$\alpha = \frac{k}{\rho C} = \frac{0.84}{1700 \times 800} = 6.17647 \times 10^{-7}\ \mathrm{m^2 s^{-1}}$$

For diurnal temperature variations $\omega = 2\pi \div (60 \times 60 \times 24) = 2\pi/86400$ rad s^{-1}.

We can use the identity $\sqrt{j} = (1 + j)/\sqrt{2}$ to rewrite (5.1) and (5.2) as follows:

$$M = L\sqrt{j\omega/\alpha} = L(1+j)\sqrt{\frac{\omega}{2\alpha}}$$

$$N = k\sqrt{j\omega/\alpha} = k(1+j)\sqrt{\frac{\omega}{2\alpha}}$$

The square root term is

$$\sqrt{\frac{\omega}{2\alpha}} = \sqrt{\frac{2\pi}{86400 \times 2 \times 6.17647 \times 10^{-7}}} = 7.67269$$

so

$$M = L(1+j)\sqrt{\frac{\omega}{2\alpha}} = 0.105 \times (1+j) \times 7.67269 = 0.805633(1+j)$$

and

$$N = k(1+j)\sqrt{\frac{\omega}{2\alpha}} = 0.84 \times (1+j) \times 7.67269 = 6.44506(1+j)$$

Thus

$$\begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} = \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{bmatrix} \begin{bmatrix} 0 \\ Q(L) \end{bmatrix}$$

where

$$M = 0.805633(1+j)$$

and

$$N = 6.44506(1+j)$$

Example 2

A sinusoidal temperature variation with a mean of 0°C, an amplitude of 10°C, and a period of one day is applied to the face $x = 0$ of the wall in Example 1. The face at $x = L$ [m] is maintained at 0°C. Calculate the heat flux through the face at $x = L$.

The heat flux at $x = L$ is given by (5.9):

$$q(L,t) = \mathrm{Im}\left[A_0 \frac{N}{\sinh M} e^{j\omega t}\right] \qquad (5.9)$$

where M and N were calculated in Example 1. Thus we have

$$\sinh M = \sinh[0.805633(1+j)]$$

$$= \frac{1}{2}\left[e^{0.805633(1+j)} - e^{-0.805633(1+j)}\right]$$

$$= \frac{1}{2}\left[e^{0.805633} e^{0.805633j} - e^{-0.805633} e^{-0.805633j}\right]$$

$$= 1.11906 e^{0.805633j} - 0.223403 e^{-0.805633j}$$

$$= 1.11906(\cos 0.805633 + j \sin 0.805633) - 0.223403(\cos 0.805633 - j \sin 0.805633)$$

$$= 0.620379 + 0.968274j$$

and

$$\frac{N}{\sinh M} = \frac{6.44506(1+j)}{0.620379 + 0.968274j}$$

$$= \frac{(0.620379 - 0.968274j)(6.44506 + 6.44506j)}{(0.620379 - 0.968274j)(0.620379 + 0.968274j)}$$

$$= \frac{3.99838 + 3.99838j - 6.24059j - 6.24059j^2}{0.384870 - 0.937554j^2}$$

$$= \frac{10.2390 - 2.24221j}{1.32242}$$

$$= 7.74257 - 1.69553j$$

The complex number $N/\sinh M$ can be represented in the complex plane as shown in Figure 4. The amplitude of $N/\sinh M$ is

$$\text{Amplitude} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \sqrt{(7.74257)^2 + (-1.69553)^2} = 7.92605$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation at $x = L$ will lag the temperature variation at $x = L$, so the phase of the heat flux variation will be negative relative to the temperature variation. Measuring the phase from the positive Real axis gives

$$\text{Phase} = -0.215584 \text{ rad } (= -12.3521°)$$

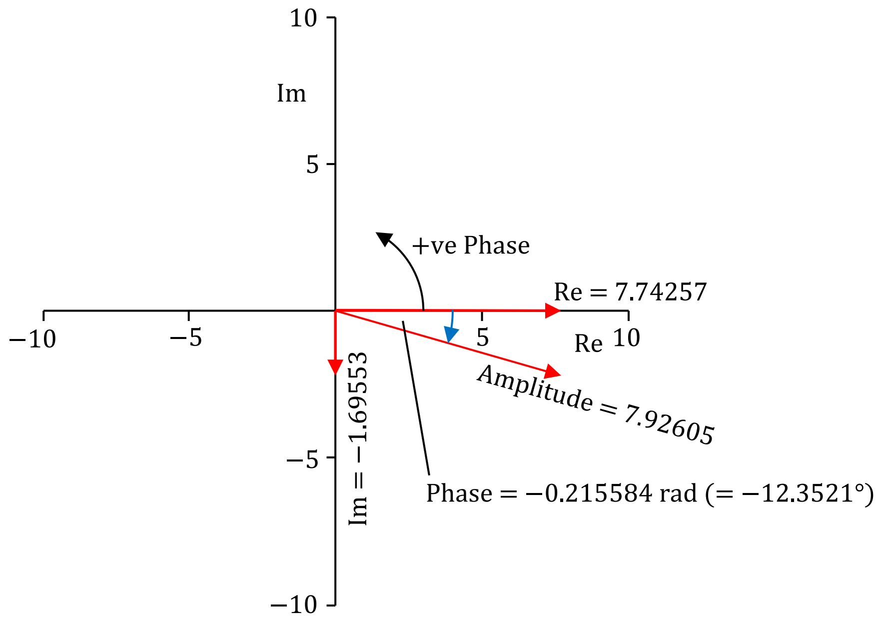
We can now write $N/\sinh M$ as

$$N/\sinh M = 7.92605[\cos(-0.215584) + j\sin(-0.215584)]$$

$$= 7.92605(\cos 0.215584 - j\sin 0.215584)$$

$$= 7.92605e^{-j0.215584} \qquad (E2.1)$$

Figure 4 Amplitude and phase of $N/\sinh M$



Substituting (E2.1) into (5.9) gives

$$q(L,t) = \text{Im}\left[A_0 7.92605e^{-0.215584j}e^{j\omega t}\right]$$

$$= \text{Im}\left[A_0 7.92605e^{(\omega t - 0.215584)j}\right]$$

$$= \text{Im}[A_0 7.92605(\cos(\omega t - 0.215584) + j\sin(\omega t - 0.215584))]$$

$$= A_0 7.92605\sin(\omega t - 0.215584)$$

Thus the amplitude of the heat flux at $x = L$ is A_0 7.92605 = 79.2605 W m^{-2}. The peak heat flux at $x = L$ lags the peak temperature at $x = 0$ by 0.215584 rad (= 12.3521°). In terms of hours, the lag is 24 hr × 12.3521°/360° = 0.823473 hr (49 min).

Example 3

Repeat Examples 1 and 2, but with the specific heat capacity of the bricks reduced to 80 J kg^{-1} K^{-1}.

With the specific heat capacity of the bricks reduced to 80 J kg^{-1} K^{-1}, the diffusivity α [m^2 s^{-1}] of the bricks is

$$\alpha = \frac{k}{\rho C} = \frac{0.84}{1700 \times 80} = 6.17647 \times 10^{-6}\ \mathrm{m^2 s^{-1}}$$

Substituting into (5.1) and (5.2),

$$M = L\sqrt{j\omega/\alpha} = 0.105\sqrt{j2\pi/(86400 \times 6.17647 \times 10^{-6})}$$

$$= 0.360290\sqrt{j} = 0.360290\frac{1}{\sqrt{2}}(1+j) = 0.2547635(1+j)$$

$$N = k\sqrt{j\omega/\alpha} = 0.84\sqrt{j2\pi/(86400 \times 6.17647 \times 10^{-6})}$$

$$= 2.88232\sqrt{j} = 2.88232\frac{1}{\sqrt{2}}(1+j) = 2.03811(1+j)$$

The instantaneous heat flux through the face at $x = L$ is given by (5.9)

$$q(L,t) = \mathrm{Im}\left[A_0 \frac{N}{\sinh M} e^{j\omega t}\right]$$

We now have

$$\sinh M = \sinh[0.2547635(1+j)]$$

$$= \frac{1}{2}\left[e^{0.2547635(1+j)} - e^{-0.2547635(1+j)}\right]$$

$$= \frac{1}{2}\left[e^{0.2547635}e^{0.2547635j} - e^{-0.2547635}e^{-0.2547635j}\right]$$

$$= 0.645078e^{0.2547635j} - 0.387550e^{-0.2547635j}$$

$$= 0.645078(\cos 0.2547635 + j\sin 0.2547635) - 0.387550(\cos 0.2547635 - j\sin 0.2547635)$$

$$= 0.249216 + 0.260239j$$

Thus

$$\frac{N}{\sinh M} = \frac{2.03811(1+j)}{0.249216 + 0.260239j}$$

$$= \frac{(0.249216 - 0.260239j)(2.03811 + 2.03811j)}{(0.249216 - 0.260239j)(0.249216 + 0.260239j)}$$

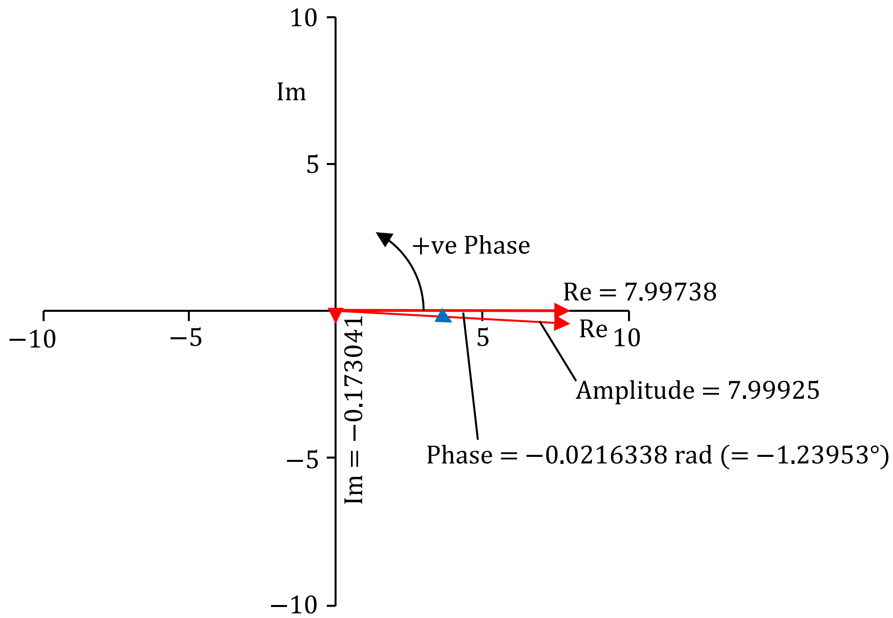
$$= \frac{0.507929 + 0.507929j - 0.530396j - 0.530396j^2}{0.0621087 - 0.0677245j^2}$$

$$= \frac{1.03833 - 0.0224665j}{0.129833}$$

$$= 7.99738 - 0.173041j$$

The complex number $N/\sinh M$ can be represented in the complex plane as shown in Figure 5.

Figure 5 Amplitude and phase of $N/\sinh M$



The amplitude of $N/\sinh M$ is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{(7.99738)^2 + (-0.173041)^2} = 7.99925$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation at $x = L$ will lag the temperature variation at $x = L$, so the phase of the heat flux variation will be negative relative to the temperature variation. Measuring the phase from the positive Real axis gives

$$\text{Phase} = -0.0216338 \text{ rad } (= -1.23953°)$$

We can now write $N/\sinh M$ as

$$N/\sinh M = 7.99925[\cos(-0.0216338) + j\sin(-0.0216338)]$$

$$= 7.99925(\cos 0.0216338 - j\sin 0.0216338)$$

$$= 7.99925e^{-j0.0216338} \qquad \text{(E3.1)}$$

Substituting (E3.1) into (5.9) gives

$$q(L,t) = \mathrm{Im}\left[A_0 7.99925e^{-0.0216338j}e^{j\omega t}\right]$$

$$= \mathrm{Im}\left[A_0 7.99925e^{(\omega t-0.0216338)j}\right]$$

$$= \mathrm{Im}[A_0 7.99925(\cos(\omega t - 0.0216338) + j\sin(\omega t - 0.0216338))]$$

$$= A_0 7.99925\sin(\omega t - 0.0216338)$$

Thus the amplitude of the heat flux at $x = L$ is A_0 $7.99925 = 79.9925$ W m^{-2}. The peak heat flux at $x = L$ lags the peak temperature at $x = 0$ by 0.0216338 rad (= 1.23953°). In terms of hours, the lag is 24 hr × $1.23953°/360°$ = 0.0826353 hr (5 min). The reduction in the specific heat capacity from 800 J kg^{-1} K^{-1} to 80 J kg^{-1} K^{-1} has reduced the lag from 49 min to 5 min.

Example 4

Calculate the heat flux through the face at $x = 0$ in Example 2.

The instantaneous heat flux through the face at $x = 0$ is given by Equation (5.8):

$$q(0,t) = \mathrm{Im}\left[A_0 \frac{N}{\tanh M} e^{j\omega t}\right] \qquad (5.8)$$

We have

$$\tanh M = \tanh[0.805633(1+j)]$$

$$= \frac{\sinh[0.805633(1+j)]}{\cosh[0.805633(1+j)]}$$

$$= \frac{\frac{1}{2}\left[e^{0.805633(1+j)} - e^{-0.805633(1+j)}\right]}{\frac{1}{2}\left[e^{0.805633(1+j)} + e^{-0.805633(1+j)}\right]}$$

$$= \frac{e^{0.805633}e^{0.805633j} - e^{-0.805633}e^{-0.805633j}}{e^{0.805633}e^{0.805633j} + e^{-0.805633}e^{-0.805633j}}$$

$$= \frac{2.23811e^{0.805633j} - 0.446805e^{-0.805633j}}{2.23811e^{0.805633j} + 0.446805e^{-0.805633j}}$$

$$= \frac{2.23811(\cos 0.805633 + j\sin 0.805633) - 0.446805(\cos 0.805633 - j\sin 0.805633)}{2.23811(\cos 0.805633 + j\sin 0.805633) + 0.446805(\cos 0.805633 - j\sin 0.805633)}$$

$$= \frac{1.24076 + 1.93655j}{1.85972 + 1.29201j}$$

$$= \frac{(1.85972 - 1.29201j)(1.24076 + 1.93655j)}{(1.85972 - 1.29201j)(1.85972 + 1.29201j)}$$

$$= \frac{2.30746 + 3.60144j - 1.60308j - 2.50205j^2}{3.45856 - 1.66930j^2}$$

$$= \frac{4.80951 + 1.99836}{5.12786}$$

$$= 0.937917 + 0.389707j$$

Thus

$$\frac{N}{\tanh M} = \frac{6.44506(1+j)}{0.937917 + 0.389707j}$$

$$= \frac{(0.937917 - 0.389707j)(6.44506 + 6.44506j)}{(0.937917 - 0.389707j)(0.937917 + 0.389707j)}$$

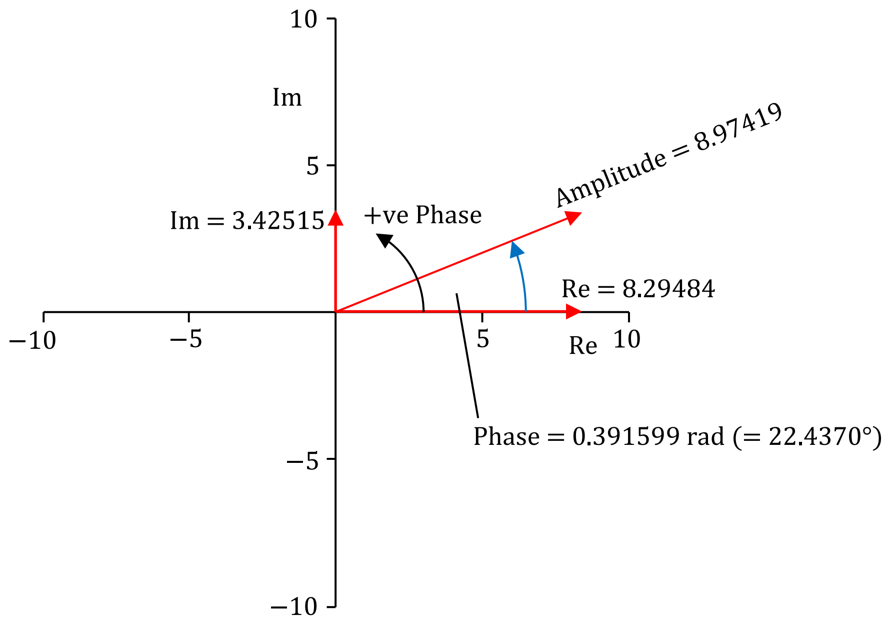
$$= \frac{6.04493 + 6.04493j - 2.51168j - 2.51168j^2}{0.879688 - 0.151871j^2}$$

$$= \frac{8.55662 + 3.53325j}{1.03156}$$

$$= 8.29484 + 3.42515j$$

The complex number $N/\tanh M$ can be represented in the complex plane as shown in Figure 6.

Figure 6 Amplitude and phase of $N/\tanh M$



The amplitude of $N/\tanh M$ is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{(8.29484)^2 + (3.42515)^2} = 8.97419$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation at $x = 0$ will lead the temperature variation at $x = 0$, so the phase of the heat flux variation will be positive relative to the temperature variation. Measuring the phase from the positive Real axis gives

$$\text{Phase} = 0.391599 \text{ rad } (= 22.4370°)$$

We can now write $N/\tanh M$ as

$$N/\tanh M = 8.97419[\cos(0.391599) + j\sin(0.391599)]$$

$$= 8.97419e^{0.391599j} \qquad \text{(E4.1)}$$

Substituting (E4.1) into (5.8) gives

$$q(0,t) = \mathrm{Im}\left[A_0 8.97419e^{0.391599j}e^{j\omega t}\right]$$

$$= \mathrm{Im}\left[A_0 8.97419e^{(\omega t+0.391599)j}\right]$$

$$= \mathrm{Im}[A_0 8.97419(\cos(\omega t + 0.391599) + j\sin(\omega t + 0.391599))]$$

$$= A_0 8.97419\sin(\omega t + 0.391599)$$

Thus the amplitude of the heat flux at $x = 0$ is A_0 $8.97419 = 89.7419$ W m^{-2}. The peak heat flux at $x = 0$ leads the peak temperature at $x = 0$ by 0.391599 rad ($= 22.4370°$). In terms of hours, the lead is 24 hr $\times$ $22.4370°/360° = 1.49580$ hr (1 hr 30 min).

6 Inverse complex transmission matrix

If we want to find the heat fluxes into and out a homogeneous slab when the temperature oscillates on the side $x = L$ and the temperature is zero on the side $x = 0$, then we need to find the inverse of the complex transmission matrix:

$$\begin{bmatrix} A_L \\ Q(L) \end{bmatrix} = \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{bmatrix}^{-1} \begin{bmatrix} A_0 \\ Q(0) \end{bmatrix}$$

The determinant of the complex transmission matrix is $\cosh^2 M - \sinh^2 M = 1$, so the inverse of the matrix can be calculated quite easily. Full details are given in the Appendix. The inverse complex transmission matrix is given by

$$\begin{bmatrix} A_L \\ Q(L) \end{bmatrix} = \begin{bmatrix} \cosh(M) & -\dfrac{\sinh(M)}{N} \\ -N \sinh(M) & \cosh(M) \end{bmatrix} \begin{bmatrix} A_0 \\ Q(0) \end{bmatrix} \quad (6.1)$$

We set $A_0 = 0$ in the matrix equation, so

$$\begin{bmatrix} A_L \\ Q(L) \end{bmatrix} = \begin{bmatrix} \cosh(M) & -\dfrac{\sinh(M)}{N} \\ -N \sinh(M) & \cosh(M) \end{bmatrix} \begin{bmatrix} 0 \\ Q(0) \end{bmatrix} \quad (6.2)$$

The matrix equation (6.2) gives

$$A_L = -\frac{\sinh(M)}{N} Q(0)$$

so

$$Q(0) = -A_L \frac{N}{\sinh(M)} \quad (6.3)$$

and

$$Q(L) = Q(0) \cosh M = -A_L \frac{N}{\tanh M} \quad (6.4)$$

Suppose the sinusoidal temperature on the side of the homogeneous slab at $x = L$ is

$$\theta(L, t) = A_L \sin(\omega t) = \mathrm{Im}\left(A_L e^{j\omega t}\right) \quad (6.5)$$

then the instantaneous heat flux through the face at $x = 0$ is

$$q(0, t) = \mathrm{Im}\left[Q(0) e^{j\omega t}\right] = \mathrm{Im}\left[-A_L \frac{N}{\sinh(M)} e^{j\omega t}\right] \quad (6.6)$$

and the instantaneous heat flux through the face at $x = L$ is

$$q(L, t) = \mathrm{Im}\left[Q(L) e^{j\omega t}\right] = \mathrm{Im}\left[-A_L \frac{N}{\tanh M} e^{j\omega t}\right] \quad (6.7)$$

Example 5

A sinusoidal temperature variation with a mean of 0°C, an amplitude of 10°C, and a period of one day is applied to the face $x = L$ of the wall in Example 1. The face at $x = 0$ is maintained at 0°C. Calculate the heat flux through (a) the face at $x = 0$ and (b) the face at $x = L$.

(a) The instantaneous heat flux through the face at $x = 0$ is given by (6.6)

$$q(0,t) = \mathrm{Im}\left[-A_L \frac{N}{\sinh(M)} e^{j\omega t}\right] \qquad (6.6)$$

We have the complex constant $N/\sinh M$ from Example 2:

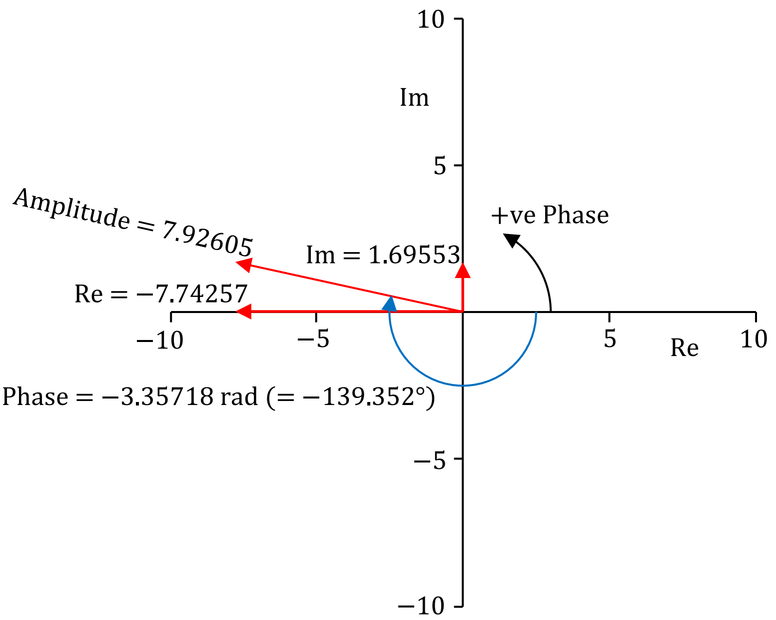
$$\frac{N}{\sinh M} = 7.74257 - 1.69553j$$

so

$$-\frac{N}{\sinh M} = -7.74257 + 1.69553j$$

The complex number $-N/\sinh M$ can be represented in the complex plane as shown in Figure 7.

Figure 7 Amplitude and phase of $-N/\sinh M$



The amplitude of $-N/\sinh M$ is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{(-7.74257)^2 + 1.69553^2} = 7.92605$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation at $x = 0$ will lag the temperature variation at $x = L$, so the phase of the heat flux variation will be negative relative to the temperature variation. Measuring the phase from the positive Real axis gives

$$\text{Phase} = -3.35718 \text{ rad } (= -192.352°)$$

We can now write $-N/\sinh M$ as

$$-N/\sinh M = 7.92605[\cos(-3.35718) + j\sin(-3.35718)]$$

$$= 7.92605e^{-3.35718j} \qquad \text{(E5.1)}$$

Substituting (E5.1) into (6.6) gives

$$q(0,t) = \text{Im}\left[A_L 7.92605e^{-3.35718j}e^{j\omega t}\right]$$

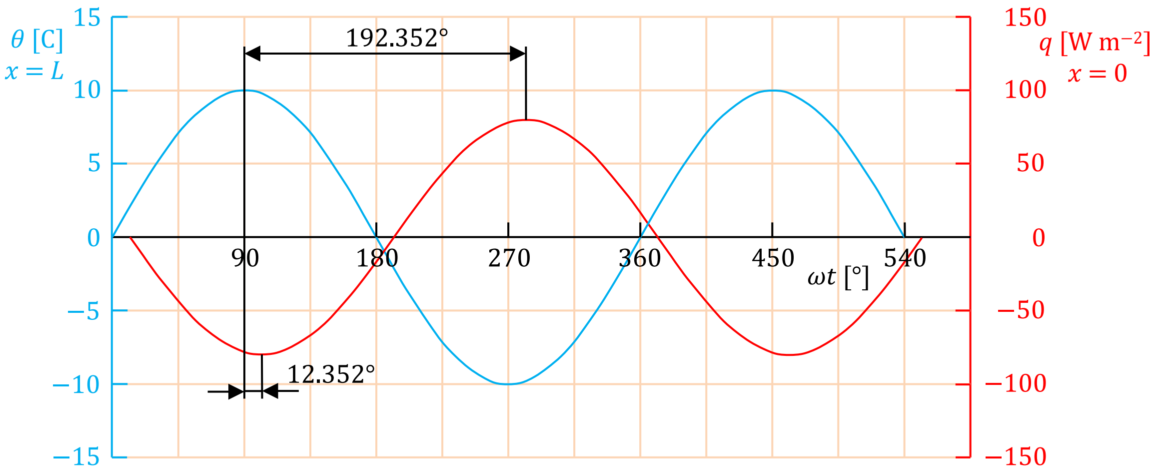
$$= \text{Im}\left[A_L 7.92605e^{j(\omega t - 3.35718)}\right]$$

$$= A_L 7.92605\sin(\omega t - 3.35718)$$

The phase lag of -3.35718 rad ($= -192.352°$) is the lag between the positive peak in the heat flux oscillations and the positive peak in the temperature oscillations. The heat flux at $x = 0$ is in the negative x direction when the temperature oscillations at $x = L$ pass through $+A_L$ and a heat flux in the negative x direction is defined to be negative. An extra $-180°$ has been added to the phase lag to go from the negative peak in the heat flux to the positive peak, as shown in Figure 8. If we remove this extra $-180°$ then the phase lag becomes $-12.352°$. As we would expect, this is the same as the phase lag that was calculated in Example 2, when the temperature oscillations were on the side $x = 0$.

The amplitude of the heat flux at $x = 0$ is A_0 $7.92605 = 79.2605$ W m^{-2} and, as we would expect, is the same as the amplitude calculated at $x = L$ in Example 2.

Figure 8 Temperature θ at $x = L$ and heat flux q at $x = 0$



(b) The instantaneous heat flux through the face at $x = L$ is given by (6.7)

$$q(L,t) = \text{Im}\left[-A_L \frac{N}{\tanh M} e^{j\omega t}\right] \qquad (6.7)$$

We have the complex constant $N/\tanh M$ from Example 4:

$$\frac{N}{\tanh M} = 8.29484 + 3.42515j$$

so

$$-\frac{N}{\tanh M} = -8.29484 - 3.42515j$$

The complex number $-N/\tanh M$ can be represented in the complex plane as shown in Figure 9.

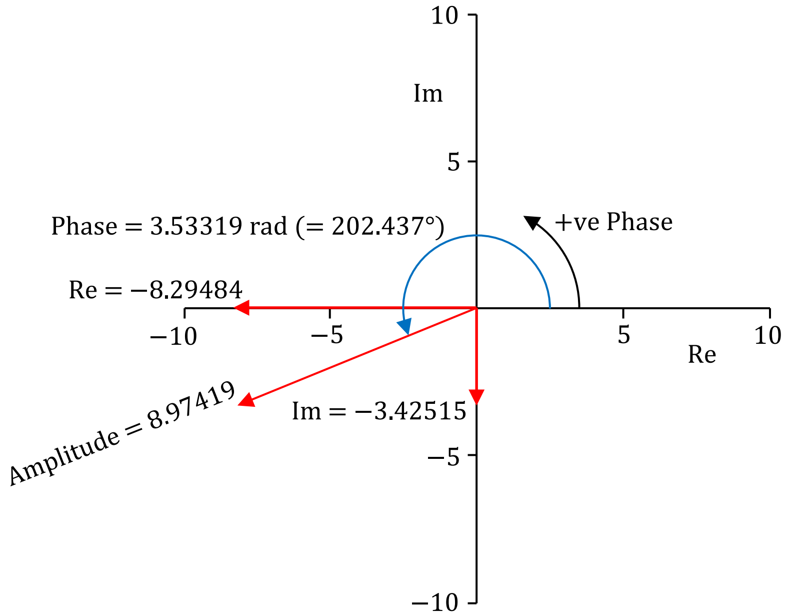
The amplitude of $-N/\tanh M$ is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{(-8.29484)^2 + (-3.42515)^2} = 8.97419$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation at $x = 0$ will lead the temperature variation at $x = L$, so the phase of the heat flux variation will be positive relative to the temperature variation. Measuring the phase from the positive Real axis gives

$$\text{Phase} = 3.53319 \text{ rad } (= 202.437°)$$

Figure 9 Amplitude and phase of $-N/\tanh M$



We can now write $-N/\tanh M$ as

$$-N/\tanh M = 8.97419[\cos 3.53319 + j\sin 3.53319]$$

$$= 8.97419e^{3.53319j} \qquad \text{(E5.2)}$$

Substituting (E5.2) into (6.7) gives

$$q(L,t) = \text{Im}\left[A_L 8.97419e^{3.53319j}e^{j\omega t}\right]$$

$$= \text{Im}\left[A_L 8.97419e^{j(\omega t+3.53319)}\right]$$

$$= A_L 8.97419\sin(\omega t + 3.53319)$$

The phase lead of 3.53319 rad (= 202.437°) is the lead between the positive peak in the heat flux oscillations and the positive peak in the temperature oscillations. The heat flux at $x = L$ is in the negative x direction when the temperature oscillations at $x = L$ pass through $+A_L$ and a heat flux in the negative x direction is defined to be negative. An extra 180° has been added to the phase lag to go from the negative peak in the heat flux to the positive peak. If we remove this extra 180° then the phase lead becomes 22.437°. As we would expect, this is the same as the phase lead that was calculated in Example 4, when the temperature oscillations were on the side $x = 0$.

The amplitude of the heat flux at $x = L$ is A_L 8.97419 = 89.7419 W m^{-2} and, as we would expect, is the same as the amplitude calculated at $x = 0$ in Example 4.

7 Temperature excitation on both sides of the slab

We can use the principle of superposition to calculate the heat fluxes into and out of the slab when we have temperature oscillations on both sides. When the temperature oscillations are at $x = 0$, we have

at $x = 0$,

$$\theta(0,t) = A_0 \sin(\omega t) = \mathrm{Im}\left(A_0 e^{j\omega t}\right) \qquad (5.4)$$

$$q(0,t) = \mathrm{Im}\left[A_0 \frac{N}{\tanh M} e^{j\omega t}\right] \qquad (5.8)$$

and at $x = L$,

$$\theta(L,t) = 0$$

$$q(L,t) = \mathrm{Im}\left[A_0 \frac{N}{\sinh M} e^{j\omega t}\right] \qquad (5.9)$$

where

$$M = L\sqrt{j\omega/\alpha}, \quad N = k\sqrt{j\omega/\alpha}$$

When the temperature oscillations are at $x = L$, we have

at $x = 0$,

$$\theta(0,t) = 0$$

$$q(0,t) = \mathrm{Im}\left[-A_L \frac{N}{\sinh M} e^{j\omega t}\right] \qquad (6.6)$$

and at $x = L$,

$$\theta(L,t) = A_L \sin(\omega t) = \mathrm{Im}\left(A_L e^{j\omega t}\right) \qquad (6.5)$$

$$q(L,t) = \mathrm{Im}\left[-A_L \frac{N}{\tanh M} e^{j\omega t}\right] \qquad (6.7)$$

where, again,

$$M = L\sqrt{j\omega/\alpha}, \quad N = k\sqrt{j\omega/\alpha}$$

We can simply add together the two equations for $q(0,t)$ and the two equations for $q(L,t)$, so

at $x = 0$,

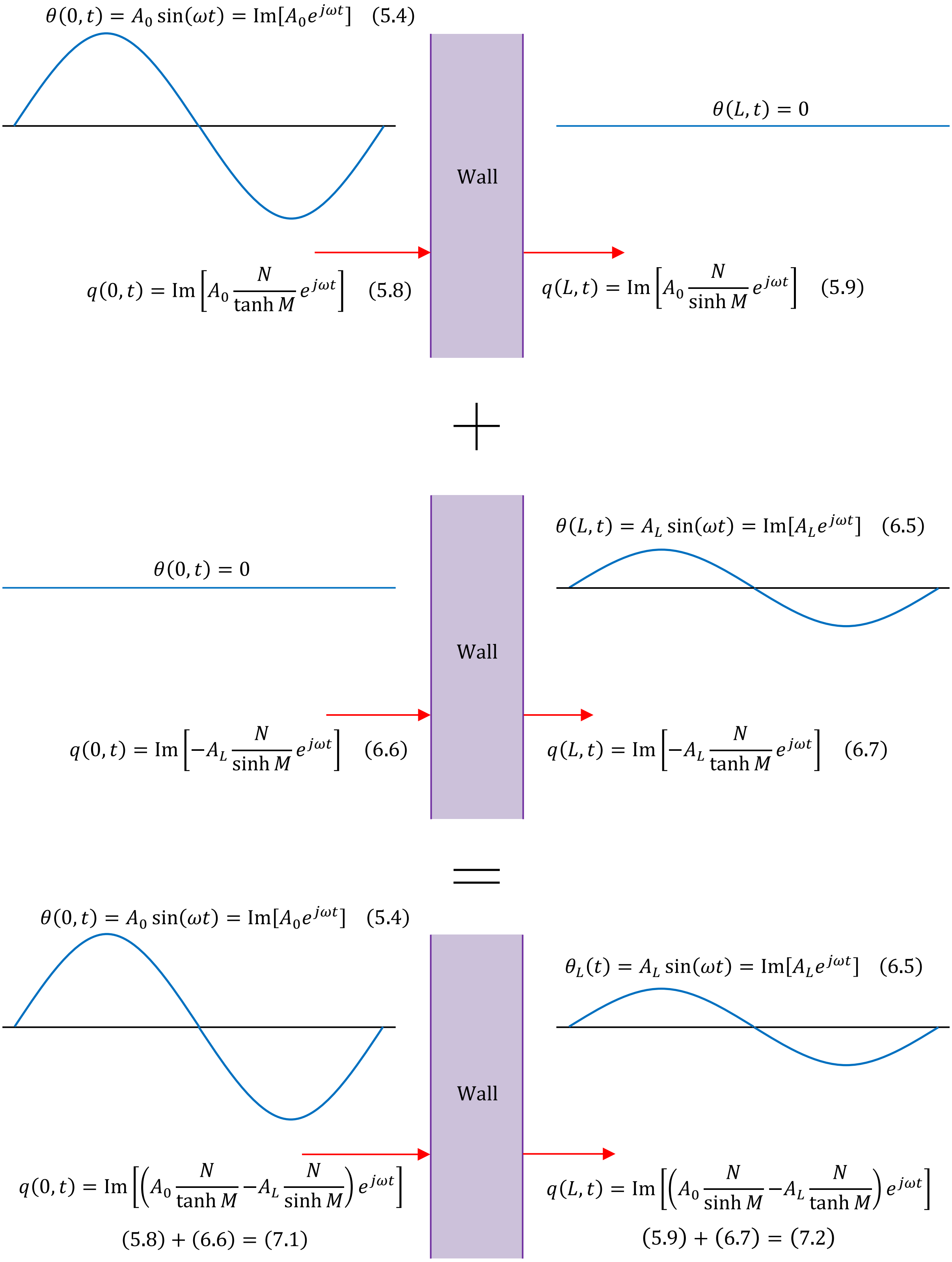
$$q(0,t) = \mathrm{Im}\left[\left(A_0 \frac{N}{\tanh M} - A_L \frac{N}{\sinh M}\right) e^{j\omega t}\right] \quad (7.1)$$

and at $x = L$,

$$q(L,t) = \mathrm{Im}\left[\left(A_0 \frac{N}{\sinh M} - A_L \frac{N}{\tanh M}\right) e^{j\omega t}\right] \quad (7.2)$$

Figure 10 summarises the calculation process.

Figure 10 Heat fluxes due to sinusoidal temperature oscillations at $x = 0$ and $x = L$



Example 6

The slab in Example 1 is subjected to a daily sinusoidal temperature variation on its surface $x = 0$ having a mean of 14°C and an amplitude of 8°C. The peak in temperature occurs at 3:00 pm. The surface $x = L$ is subjected to a daily sinusoidal temperature variation having a mean of 21°C and an amplitude of 6°C. The peak temperature occurs at 12:00 noon. Calculate:

a. the mean heat flux through the wall,
b. the heat fluxes on the two sides of the slab due to the temperature variation on the surface $x = 0$,
c. the heat fluxes on the two sides of the slab due to the temperature variation on the surface $x = L$,
d. the net heat fluxes on the two sides.

(a) We can determine the mean heat flux from Fourier's law:

$$q(x,t) = -k\frac{\partial\theta(x,t)}{\partial x}$$

Integrating with respect t gives

$$\bar{q}(x) = -\frac{k}{t_2 - t_1}\int_{t_1}^{t_2}\frac{\partial\theta(x,t)}{\partial x}dt = -k\frac{d\bar{\theta}(x)}{dx} \quad \text{(E6.1)}$$

where the overbar represents a time average. The time-average heat flux into and out of any layer of the slab must be the same otherwise the temperature of the layer will increase or decrease continually. Consequently, $\bar{q}$ cannot be a function of x, and since k is constant, the temperature gradient $d\bar{\theta}/dx$ must be constant. Integrating (E6.1) with respect to x gives

$$q_{mean} = -\frac{k}{L}\int_{x_0}^{x_L}\frac{d\bar{\theta}(x)}{dx}dx = -\frac{k}{L}\int_{\bar{\theta}_0}^{\bar{\theta}_L}d\bar{\theta} = -k\frac{\bar{\theta}_L - \bar{\theta}_0}{L} = -k\frac{\theta_L - \theta_0}{L}$$

since θ_0 and θ_L are constant. The mean heat flux q_{mean} is therefore

$$q_{mean} = -0.84 \times \frac{21 - 14}{0.105} = -56\ \text{W m}^{-2} \quad (6.2)$$

(b) From (5.4) the temperature oscillations at $x = 0$ are given by

$$\theta(0,t) = A_0\sin(\omega t - 0.75\pi) = 8\sin(\omega t - 0.75\pi) = \text{Im}\left[8\,e^{j(\omega t - 0.75\pi)}\right] \quad \text{(E6.3)}$$

The peak in the temperature occurs at 15:00 hr, so we must subtract $(15{:}00 - 6{:}00)\times 2\pi/24$ from the argument of the sine function.

From (5.8) the heat flux at $x = 0$ is

$$q(0,t) = \text{Im}\left[A_0\frac{N}{\tanh M}e^{j(\omega t - 0.75\pi)}\right] \quad \text{(E6.4)}$$

and from (5.9) the heat flux at $x = L$ is

$$q(L,t) = \text{Im}\left[A_0\frac{N}{\sinh M}e^{j(\omega t - 0.75\pi)}\right] \quad \text{(E6.5)}$$

We calculated the complex constants $N/\tanh M$ and $N/\sinh M$ in Examples 4 and 2, respectively.

$$N/\tanh M = 8.97419e^{0.391599j} \qquad (E4.1)$$

$$N/\sinh M = 7.92605e^{-j0.215584} \qquad (E2.1)$$

Substituting (E4.1) into (E6.4) gives

$$q(0,t) = \mathrm{Im}\big[A_0 8.97419e^{0.391599j}e^{j(\omega t-0.75\pi)}\big]$$

$$= \mathrm{Im}\big[8 \times 8.97419e^{j(\omega t-0.75\pi+0.391599)}\big]$$

$$= 71.7935\sin(\omega t - 0.75\pi + 0.391599) \qquad (E6.6)$$

and substituting (E2.1) into (E6.5) gives

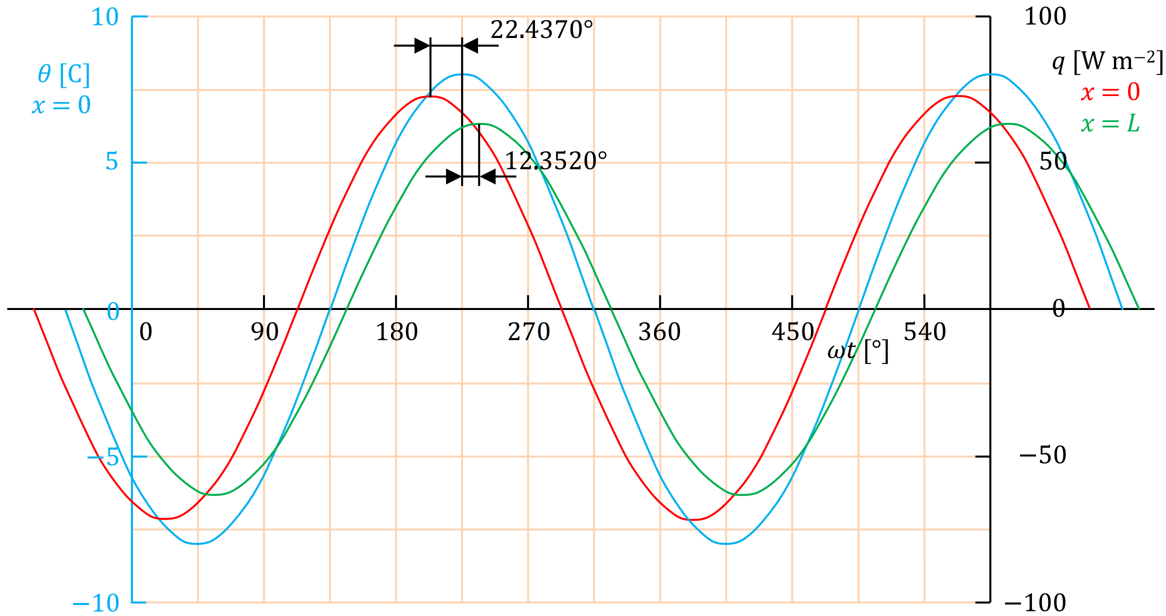
$$q(L,t) = \mathrm{Im}\big[A_0 7.92605e^{-j0.215584}e^{j(\omega t-0.75\pi)}\big]$$

$$= \mathrm{Im}\big[8 \times 7.92605e^{j(\omega t-0.75\pi-0.215584)}\big]$$

$$= 63.4084\sin(\omega t - 0.75\pi - 0.215584) \qquad (E6.7)$$

Equations (E6.3), (E6.6) and (E6.7) are plotted in Figure 11. The heat flux at $x = 0$ leads the temperature by 0.391599 rad (= 22.4370°), as expected, and the heat flux at $x = L$ lags the temperature by 0.215584 rad (= 12.3520°), as expected.

Figure 11 Temperature θ at $x = 0$ and heat flux q at $x = 0$ and $x = L$



(c) From (6.5) the temperature oscillations at $x = L$ are given by

$$\theta(L,t) = A_L \sin(\omega t - 0.5\pi) = 6\sin(\omega t - 0.5\pi) = \mathrm{Im}\left[6\, e^{j(\omega t - 0.5\pi)}\right] \quad \text{(E6.8)}$$

The peak in the temperature occurs at 12:00 hr, so we must subtract $(12{:}00 - 6{:}00)\times 2\pi/24$ from the argument of the sine function.

From (6.6) the heat flux at $x = 0$ is

$$q(0,t) = \mathrm{Im}\left[-A_L \frac{N}{\sinh M} e^{j(\omega t - 0.5\pi)}\right] \quad \text{(E6.9)}$$

and from (6.7) the heat flux at $x = L$ is

$$q(L,t) = \mathrm{Im}\left[-A_L \frac{N}{\tanh M} e^{j(\omega t - 0.5\pi)}\right] \quad \text{(E6.10)}$$

We calculated the complex constants $-N/\sinh M$ and $-N/\tanh M$ in Example 5.

$$-N/\sinh M = 7.92605 e^{-3.35718j} \quad \text{(E5.1)}$$

$$-N/\tanh M = 8.97419 e^{3.53319j} \quad \text{(E5.2)}$$

Substituting (E5.1) into (E6.9) gives

$$q(0,t) = \mathrm{Im}\left[A_L 7.92605 e^{-3.35718j} e^{j(\omega t - 0.5\pi)}\right]$$

$$= \mathrm{Im}\left[6 \times 7.92605 e^{j(\omega t - 0.5\pi - 3.35718)}\right]$$

$$= 47.5563 \sin(\omega t - 0.5\pi - 3.35718) \quad \text{(E6.11)}$$

and substituting (E5.2) into (E6.10) gives

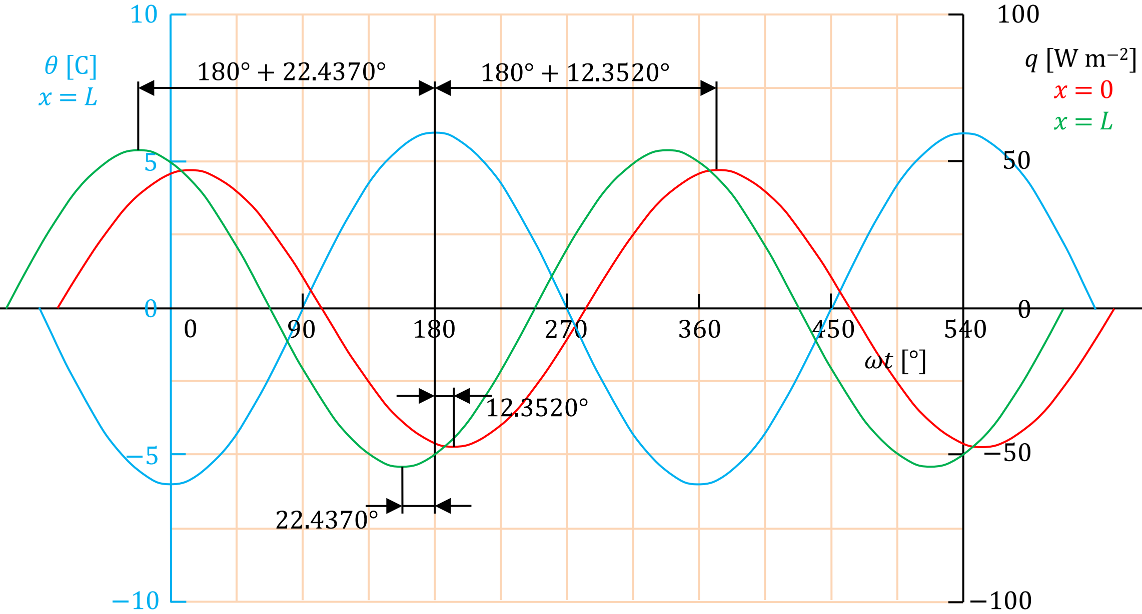
$$q(L,t) = \mathrm{Im}\left[A_L 8.97419 e^{3.53319j} e^{j(\omega t - 0.5\pi)}\right]$$

$$= \mathrm{Im}\left[6 \times 8.97419 e^{j(\omega t - 0.5\pi + 3.53319)}\right]$$

$$= 53.8451 \sin(\omega t - 0.5\pi + 3.533199) \quad \text{(E6.12)}$$

Equations (E6.8), (E6.11) and (E6.12) are plotted in Figure 12. The heat flux at $x = 0$ lags the temperature by 3.35718 rad ($= 192.3520° = 180° + 12.3520°$), as expected, and the heat flux at $x = L$ leads the temperature by 3.53319 rad ($= 202.4370° = 180° + 22.4370°$), as expected.

Figure 12 Temperature θ at $x = L$ and heat flux q at $x = 0$ and $x = L$



(d) The net heat flux at the surface $x = 0$ is the sum of (E6.2), (E6.6) and (E6.11):

$$q(0, t) = -56 + 71.7935\sin(\omega t - 0.75\pi + 0.391599)$$

$$+ 47.5563\sin(\omega t - 0.5\pi - 3.35718) \qquad \text{(E6.13))}$$

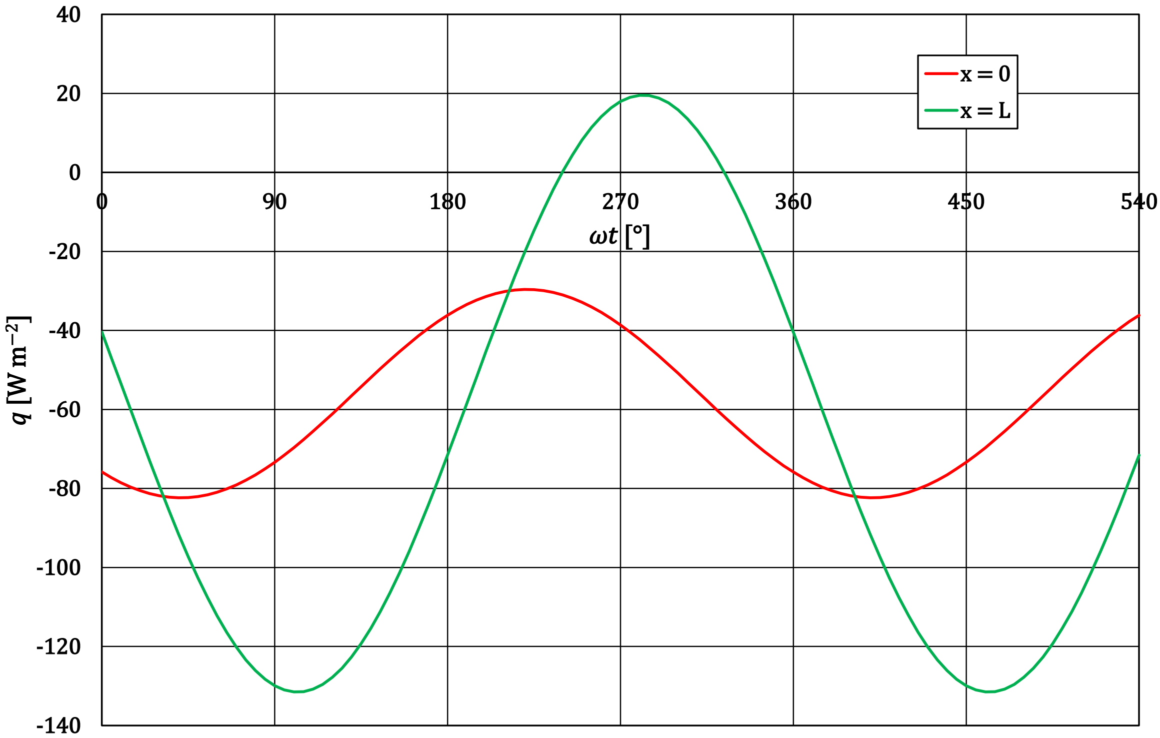
The net heat flux at the surface $x = L$ is the sum of (E6.2), (E6.7) and (E6.12):

$$q(L, t) = -56 + 63.4084\sin(\omega t - 0.75\pi - 0.215584)$$

$$+ 53.8451\sin(\omega t - 0.5\pi + 3.533199) \qquad \text{(E6.14))}$$

The net heat fluxes $q(0, t)$ and $q(L, t)$ are plotted in Figure 13. The curves are consistent with Figures 11 and 12 and equations (E6.13) and (E6.14).

Figure 13 Net heat flux q at $x = 0$ and $x = L$



8 References

1. K. N. Atkinson, *Admittance Method. 1. One-Dimensional Transient Heat Conduction. Theory Guide*, Atkinson Science Limited, 2020.*
2. M. R. Spiegel, *Schaum's Outline of Advanced Mathematics for Engineers and Scientists*, McGraw-Hill, 1971.

* Download from https://atkinsonscience.co.uk/Downloads/Construction.aspx

9 Appendix

In this Appendix we describe the operations needed to calculate the inverse of a matrix. We shall assume that the reader is familiar with the definition of a matrix and the matrix operations of addition, subtraction, and multiplication. At the end of the Appendix we shall calculate the inverse of the complex transmission matrix (6.1).

9.1 Transpose of a matrix

A square matrix of order n is a square array of numbers having n rows and n columns:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad (9.1)$$

The *transpose* of the matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^T$. It is obtained by interchanging the rows and columns so that if $\boldsymbol{A} = (a_{jk})$ then $\boldsymbol{A}^T = (a_{kj})$. If a square matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} 3 & 9 & -1 \\ 7 & 4 & 3 \\ 4 & -6 & 2 \end{bmatrix}$$

then $\boldsymbol{A}^T$ is

$$A^T = \begin{bmatrix} 3 & 7 & 4 \\ 9 & 4 & -6 \\ -1 & 3 & 2 \end{bmatrix}$$

9.2 Determinant of a matrix

If $\boldsymbol{A}$ is a square matrix of order n, then the *determinant* of $\boldsymbol{A}$ or $\det(\boldsymbol{A})$ is a number which we write as

$$\det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad (9.2)$$

If $\det(\boldsymbol{A}) = 0$ then the square matrix is called a *singular matrix*. For any element a_{jk} in the determinant, we can define a new determinant of order $n - 1$, called the *minor* of a_{jk}, which we obtain by removing all the elements in the j^{th} row and the k^{th} column.

If a fourth-order determinant is

$$\det(\boldsymbol{A}) = \begin{vmatrix} 3 & 7 & -1 & 7 \\ 11 & -2 & 8 & 5 \\ 2 & 9 & 12 & 2 \\ -8 & 4 & 3 & -1 \end{vmatrix} \qquad (9.3)$$

then the minor of the element 9 in the third row and the second column of the determinant is

$$\begin{vmatrix} 3 & -1 & 7 \\ 11 & 8 & 5 \\ -8 & 3 & -1 \end{vmatrix}$$

If we multiply the minor of the element a_{jk} by $(-1)^{j+k}$ then the result is called the *cofactor* of a_{jk} and is denoted by A_{jk}. The cofactor A_{32} of the element a_{32} in the fourth-order determinant is therefore

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -1 & 7 \\ 11 & 8 & 5 \\ -8 & 3 & -1 \end{vmatrix} = -\begin{vmatrix} 3 & -1 & 7 \\ 11 & 8 & 5 \\ -8 & 3 & -1 \end{vmatrix}$$

We can calculate the value of a determinant, in either of two ways. We can take the elements in any *row*, multiply the elements by their cofactors, and then sum the results. Suppose we take the second row of the fourth-order determinant (9.3). We obtain

$$\det(\boldsymbol{A}) = \sum_{k=1}^{4} a_{2k}A_{2k}$$

$$= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24}$$

$$= 11 \times (-1)^{2+1} \times \begin{vmatrix} 7 & -1 & 7 \\ 9 & 12 & 2 \\ 4 & 3 & -1 \end{vmatrix} + (-2) \times (-1)^{2+2} \times \begin{vmatrix} 3 & -1 & 7 \\ 2 & 12 & 2 \\ -8 & 3 & -1 \end{vmatrix}$$

$$+ 8 \times (-1)^{2+3} \times \begin{vmatrix} 3 & 7 & 7 \\ 2 & 9 & 2 \\ -8 & 4 & -1 \end{vmatrix} + 5 \times (-1)^{2+4} \times \begin{vmatrix} 3 & 7 & -1 \\ 2 & 9 & 12 \\ -8 & 4 & 3 \end{vmatrix}$$

$$= -11\begin{vmatrix} 7 & -1 & 7 \\ 9 & 12 & 2 \\ 4 & 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 & 7 \\ 2 & 12 & 2 \\ -8 & 3 & -1 \end{vmatrix} - 8\begin{vmatrix} 3 & 7 & 7 \\ 2 & 9 & 2 \\ -8 & 4 & -1 \end{vmatrix} + 5\begin{vmatrix} 3 & 7 & -1 \\ 2 & 9 & 12 \\ -8 & 4 & 3 \end{vmatrix} \qquad (9.4)$$

To progress further, we must evaluate each of the four third-order determinants in (9.4). We will evaluate the determinants based on the elements in the first row.

For the first determinant we obtain

$$\begin{vmatrix} 7 & -1 & 7 \\ 9 & 12 & 2 \\ 4 & 3 & -1 \end{vmatrix} = \sum_{k=1}^{3} a_{1k}A_{1k}$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 7 \times (-1)^{1+1} \times \begin{vmatrix} 12 & 2 \\ 3 & -1 \end{vmatrix} + (-1) \times (-1)^{1+2} \times \begin{vmatrix} 9 & 2 \\ 4 & -1 \end{vmatrix} + 7 \times (-1)^{1+3} \times \begin{vmatrix} 9 & 12 \\ 4 & 3 \end{vmatrix}$$

The second-order determinants are:

$$\begin{vmatrix} 12 & 2 \\ 3 & -1 \end{vmatrix} = 12 \times (-1)^{1+1} \times (-1) + 2 \times (-1)^{1+2} \times 3 = -18$$

$$\begin{vmatrix} 9 & 2 \\ 4 & -1 \end{vmatrix} = 9 \times (-1)^{1+1} \times (-1) + 2 \times (-1)^{1+2} \times 4 = -17$$

$$\begin{vmatrix} 9 & 12 \\ 4 & 3 \end{vmatrix} = 9 \times (-1)^{1+1} \times 3 + 12 \times (-1)^{1+2} \times 4 = -21$$

so

$$\begin{vmatrix} 7 & -1 & 7 \\ 9 & 12 & 2 \\ 4 & 3 & -1 \end{vmatrix} = 7 \times (-1)^{1+1} \times (-18) + (-1) \times (-1)^{1+2} \times (-17) + 7 \times (-1)^{1+3} \times (-21)$$

$$= -290$$

In a similar way, we can obtain the values of the other third-order determinants:

$$\begin{vmatrix} 3 & -1 & 7 \\ 2 & 12 & 2 \\ -8 & 3 & -1 \end{vmatrix} = 674$$

$$\begin{vmatrix} 3 & 7 & 7 \\ 2 & 9 & 2 \\ -8 & 4 & -1 \end{vmatrix} = 411$$

$$\begin{vmatrix} 3 & 7 & -1 \\ 2 & 9 & 12 \\ -8 & 4 & 3 \end{vmatrix} = -857$$

Substituting the values of the third-order determinants into (10.4) gives

$$\det(\boldsymbol{A}) = -11 \times (-290) - 2 \times 674 - 8 \times 411 + 5 \times (-857)$$

$$= -5731$$

In the second method of evaluating the determinant, we can take the elements in any *column*, multiply the elements by their cofactors, and then sum the results.

Suppose we take the third column of the fourth-order determinant (9.3). We obtain

$$\det(\boldsymbol{A}) = \sum_{j=1}^{4} a_{j3} A_{j3}$$

$$= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} + a_{43}A_{43}$$

$$= -1 \times (-1)^{1+3} \times \begin{vmatrix} 11 & -2 & 5 \\ 2 & 9 & 2 \\ -8 & 4 & -1 \end{vmatrix} + 8 \times (-1)^{2+3} \times \begin{vmatrix} 3 & 7 & 7 \\ 2 & 9 & 2 \\ -8 & 4 & -1 \end{vmatrix}$$

$$+12 \times (-1)^{3+3} \times \begin{vmatrix} 3 & 7 & 7 \\ 11 & -2 & 5 \\ -8 & 4 & -1 \end{vmatrix} + 3 \times (-1)^{4+3} \times \begin{vmatrix} 3 & 7 & 7 \\ 11 & -2 & 5 \\ 2 & 9 & 2 \end{vmatrix}$$

$$= -1 \times 241 - 8 \times 411 + 12 \times (-61) - 3 \times 490$$

$$= -5731$$

as expected.

9.3 Unit matrix

The *unit matrix* $\boldsymbol{I}$ is a square matrix in which all the elements of the principal diagonal are equal to 1 while all the other elements are zero. The fourth-order unit matrix is

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

9.4 Inverse of a matrix

If $\boldsymbol{A}$ is a square matrix of order n and is non-singular (i.e. $\det(\boldsymbol{A}) \neq 0$), then there exists a unique *inverse* $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\,\boldsymbol{A} = \boldsymbol{I}$ and we can determine $\boldsymbol{A}^{-1}$ from

$$\boldsymbol{A}^{-1} = \frac{\left(A_{jk}\right)^T}{\det(\boldsymbol{A})}$$

where (A_{jk}) is the matrix of the cofactors A_{jk} of $\boldsymbol{A}$ and $(A_{jk})^T = A_{kj}$ is its transpose.

Consider the third-order square matrix:

$$\boldsymbol{A} = \begin{bmatrix} 9 & -3 & 1 \\ 2 & 11 & 7 \\ -1 & 2 & -5 \end{bmatrix}$$

The determinant of the matrix $\det(\boldsymbol{A})$ is

$$\det(\boldsymbol{A}) = \sum_{k=1}^{3} a_{1k}A_{1k} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 9 \times (-1)^{1+1} \times \begin{vmatrix} 11 & 7 \\ 2 & -5 \end{vmatrix} - 3 \times (-1)^{1+2} \times \begin{vmatrix} 2 & 7 \\ -1 & -5 \end{vmatrix} + 1 \times (-1)^{1+3} \times \begin{vmatrix} 2 & 11 \\ -1 & 2 \end{vmatrix}$$

$$9 \times (-69) + 3 \times (-3) + 1 \times 15$$

$$= -615$$

The matrix of the cofactors (A_{jk}) is

$$\left(A_{jk}\right) = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 11 & 7 \\ 2 & -5 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 2 & 7 \\ -1 & -5 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 2 & 11 \\ -1 & 2 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} -3 & 1 \\ 2 & -5 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 9 & 1 \\ -1 & -5 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 9 & -3 \\ -1 & 2 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} -3 & 1 \\ 11 & 7 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 9 & 1 \\ 2 & 7 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 9 & -3 \\ 2 & 11 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -69 & 3 & 15 \\ -13 & -44 & -15 \\ -32 & -61 & 105 \end{bmatrix}$$

The inverse of $\boldsymbol{A}$ is

$$\boldsymbol{A}^{-1} = \frac{\left(A_{jk}\right)^T}{\det(\boldsymbol{A})}$$

$$= \frac{-1}{615}\begin{bmatrix} -69 & -13 & -32 \\ 3 & -44 & -61 \\ 15 & -15 & 105 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{69}{615} & \frac{13}{615} & \frac{32}{615} \\ \frac{-3}{615} & \frac{44}{615} & \frac{61}{615} \\ \frac{-15}{615} & \frac{15}{615} & \frac{-105}{615} \end{bmatrix}$$

We can check that the inverse of $\boldsymbol{A}$ has been calculated correctly by multiplying $\boldsymbol{A}^{-1}$ by $\boldsymbol{A}$:

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \begin{bmatrix} \frac{69}{615} & \frac{13}{615} & \frac{32}{615} \\ \frac{-3}{615} & \frac{44}{615} & \frac{61}{615} \\ \frac{-15}{615} & \frac{15}{615} & \frac{-105}{615} \end{bmatrix}\begin{bmatrix} 9 & -3 & 1 \\ 2 & 11 & 7 \\ -1 & 2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{69\times 9+13\times 2+32\times(-1)}{615} & \frac{69\times(-3)+13\times 11+32\times 2}{615} & \frac{69\times 1+13\times 7+32\times(-5)}{615} \\ \frac{-3\times 9+44\times 2+61\times(-1)}{615} & \frac{-3\times(-3)+44\times 11+61\times 2}{615} & \frac{-3\times 1+44\times 7+61\times(-5)}{615} \\ \frac{-15\times 9+15\times 2-105\times(-1)}{615} & \frac{-15\times(-3)+15\times 11-105\times 2}{615} & \frac{-15\times 1+15\times 7-105\times(-5)}{615} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \boldsymbol{I}$$

as expected.

9.5 Inverse of the complex transmission matrix for a uniform slab

The complex transmission matrix for a uniform slab is given by (5.3):

$$\begin{bmatrix} A_o \\ Q(0) \end{bmatrix} = \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{bmatrix} \begin{bmatrix} A_L \\ Q(L) \end{bmatrix}$$

The square matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{bmatrix}$$

so the inverse of $\boldsymbol{A}$ is

$$\boldsymbol{A}^{-1} = \frac{(A_{jk})^T}{\det(\boldsymbol{A})}$$

The determinant $\det(\boldsymbol{A})$ of $\boldsymbol{A}$ is

$$\det(\boldsymbol{A}) = \begin{vmatrix} \cosh M & \dfrac{\sinh M}{N} \\ N \sinh M & \cosh M \end{vmatrix}$$

$$= \sum_{k=1}^{2} a_{1k} A_{1k}$$

$$= \cosh M \times (-1)^{1+1} \times \cosh M + \frac{\sinh M}{N} \times (-1)^{1+2} \times N \sinh M$$

$$\cosh^2 M - \sinh^2 M$$

$$= 1$$

The matrix of cofactors (A_{jk}) is

$$(A_{jk}) = \begin{bmatrix} (-1)^{1+1} \times \cosh M & (-1)^{1+2} \times N \sinh M \\ (-1)^{2+1} \times \dfrac{\sinh M}{N} & (-1)^{2+2} \times \cosh M \end{bmatrix}$$

$$= \begin{bmatrix} \cosh M & -N \sinh M \\ -\dfrac{\sinh M}{N} & \cosh M \end{bmatrix}$$

and the transpose of the matrix of cofactors $(A_{jk})^T$ is

$$(A_{jk})^T = \begin{bmatrix} \cosh M & -\dfrac{\sinh M}{N} \\ -N \sinh M & \cosh M \end{bmatrix}$$

Finally, the inverse of $\boldsymbol{A}$ is

$$\boldsymbol{A}^{-1} = \frac{\left(A_{jk}\right)^T}{\det(\boldsymbol{A})}$$

$$= \frac{1}{1}\begin{bmatrix} \cosh M & -\dfrac{\sinh M}{N} \\ -N \sinh M & \cosh M \end{bmatrix}$$

$$= \begin{bmatrix} \cosh M & -\dfrac{\sinh M}{N} \\ -N \sinh M & \cosh M \end{bmatrix}$$